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Improving the Weizsäcker-Williams approximation in electron-proton collisions

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Abstract

We critically examine the validity of the Weizsäcker-Williams approximation in electron-hadron collisions. We show that in its commonly used form it can lead to large errors, and we show how to improve it in order to get accurate results. In particular, we present an improved form that is valid beyond the leading logarithmic approximation in the case when a small-angle cut is applied to the scattered electron. Furthermore we include comparisons of the approximate expressions with the exact electroproduction calculation in the case of heavy-quark production.

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Electroproduction phenomena are usually computed by assuming that the incoming electron beam can be considered to be equivalent to a photon broad-band beam [1]. The photon distribution in the electron is given by the formula

$$f_\gamma^{(e)}(y) = \frac{\alpha_{\rm em}}{2\pi}\left[\frac{1+(1-y)^2}{y}\log\frac{(Sy-S_{min})(1-y)}{m_e^2 y^2} + \mathcal{O}(1)\right], \tag{1}$$

where S is the electron-hadron centre-of-mass energy squared, S_{min} is the minimum value for the invariant mass squared of the produced hadronic system, m_e is the electron mass, and y is the fraction of the electron energy carried by the photon. Several examples of applications of this formula, as well as discussions of its range of validity, can be found in ref. [2]. We have explicitly indicated in eq. (1) that the neglected terms are of order 1 with respect to the logarithmically-enhanced one. The logarithmic term is typically of order 15 to 20 at HERA energies, and therefore one expects the error associated with this approximation to be of the order of 5 to 10%. It was noted in ref. [3] that the error can in fact be much larger than this, unless one takes appropriately into account the dynamical effects that modify the "effective" upper scale entering the logarithm in eq. (1).

In this paper, we show that several modifications to eq. (1) are needed in order to meet the needs of the photoproduction experimental conditions one encounters at HERA. There a photoproduction event is usually defined with an appropriate anti-tag condition (i.e. outgoing electrons above a given angular cut are vetoed). Therefore, we reconsider the derivation of the Weizsäcker-Williams distribution, taking into account the particular experimental conditions of HERA. First of all, we derive a Weizsäcker-Williams formula for the case when a small-angle cut is applied to the outgoing electron, which is valid beyond the leading logarithmic approximation.

Let us consider the electroproduction process

$$e(p) + p(k) \to e(p') + X, \tag{2}$$

where p is a massless parton ($k^2 = 0$), X is a generic hadronic system, and $p^2 = p'^2 = m_e^2$. The cross section for the process is given by

$$d\sigma_{ep} = \frac{1}{8k\cdot p}\frac{e^2 W^{\mu\nu}T_{\mu\nu}}{q^4}\frac{d^3p'}{(2\pi)^3 2E'}\ , \tag{3}$$

where

$$q = p - p'. \tag{4}$$

The tensor $T_{\mu\nu}$ is the electron tensor, given by

$$T_{\mu\nu} = 4\left(\frac{1}{2}q^2 g_{\mu\nu} + p_\mu p'_\nu + p_\nu p'_\mu\right). \tag{5}$$

Notice that

$$q^\mu T_{\mu\nu} = q^\nu T_{\mu\nu} = 0, \tag{6}$$

since $p \cdot q = -p' \cdot q = q^2/2$.

The tensor $W^{\mu\nu}$ is the hadron tensor. Exploiting current conservation, which requires $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$, it can be decomposed as

$$W^{\mu\nu} = W_1(q^2, k{\cdot}q)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) - \frac{q^2}{(k\cdot q)^2} W_2(q^2, k{\cdot}q)\left(k^\mu - \frac{k\cdot q}{q^2}q^\mu\right)\left(k^\nu - \frac{k\cdot q}{q^2}q^\nu\right) \tag{7}$$

(a term proportional to $\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma$ has not been included, since it gives zero contribution in this case). In the limit $q^2 \to 0$, $W^{\mu\nu}$ must be an analytic function of q^2. By requiring that $q^2 W^{\mu\nu}$ vanish for $q^2 = 0$, we obtain

$$W_2(q^2, k\cdot q) = W_1(0, k\cdot q) + \mathcal{O}(q^2). \tag{8}$$

We drop for the moment the $\mathcal{O}(q^2)$ terms. We will discuss later their effect, which in general leads to non-universal and non-logarithmic corrections to the factorised Weizsäcker-Williams approximation. From eqs. (5), (7) and (8) we find

$$W^{\mu\nu}T_{\mu\nu} = -4W_1(0, k\cdot q)\left[2m_e^2 + q^2\frac{1+(1-y)^2}{y^2}\right], \tag{9}$$

where

$$y = \frac{k\cdot q}{k\cdot p} = 1 - \frac{k\cdot p'}{k\cdot p}. \tag{10}$$

Notice that y is precisely the fraction z of longitudinal momentum carried by the photon. In fact, if $k = k^0(1,0,0,-1)$ and $q = zp + \hat{q}$, with $k\cdot\hat{q} = 0$ one finds $y = (k\cdot q)/(k\cdot p) = z$.

We must now find a convenient expression for the phase space of the scattered electron. We have

$$\frac{d^3p'}{E'} = 2\pi\beta'^2 E' dp' d\cos\theta, \tag{11}$$

where the trivial azimuth integration has already been carried out, and

$$p' = (E', 0, E'\beta'\sin\theta, E'\beta'\cos\theta) \tag{12}$$

$$p = (E, 0, 0, E\beta) \tag{13}$$

$$\beta' = \sqrt{1 - \frac{m_e^2}{E'^2}}, \quad \beta = \sqrt{1 - \frac{m_e^2}{E^2}}. \tag{14}$$

We now trade the two variables $(p', \cos\theta)$ for (q^2, y). From the definitions given above, we find

$$q^2 = 2m_e^2 - 2EE'\,(1 - \beta\beta'\cos\theta) \tag{15}$$

$$y = 1 - \frac{E'\,(1 + \beta'\cos\theta)}{E(1+\beta)}, \tag{16}$$

and it is easy to prove that the Jacobian of this change of variables is simply $2E'\beta'^2$. Therefore

$$\frac{d^3p'}{E'} = \pi dq^2 dy. \tag{17}$$

We are now ready to compute the cross section in eq. (3). We get

$$d\sigma_{ep} = -\frac{\alpha_{\rm em}}{2\pi}\frac{W_1(0, k\cdot q)}{4k\cdot p}\left[\frac{2m_e^2}{q^4} + \frac{1+(1-y)^2}{y^2q^2}\right] dq^2 dy, \tag{18}$$

where, as usual, $\alpha_{\rm em} = e^2/(4\pi)$. Integration in q^2 gives

$$d\sigma_{ep} = \sigma_{\gamma p}(q, k) f_\gamma^{(e)}(y) dy, \tag{19}$$

where

$$f_\gamma^{(e)}(y) = \frac{\alpha_{\rm em}}{2\pi}\left[2m_e^2 y\left(\frac{1}{q^2_{max}} - \frac{1}{q^2_{min}}\right) + \frac{1+(1-y)^2}{y}\log\frac{q^2_{min}}{q^2_{max}}\right], \tag{20}$$

and

$$\sigma_{\gamma p}(q, k) = -\frac{g_{\mu\nu}W^{\mu\nu}}{8k\cdot q} = \frac{W_1(0, k\cdot q)}{4k\cdot q} \tag{21}$$

is the cross section for the process $\gamma(q) + p(k) \to X$ for an on-shell photon.

We must determine the two integration bounds q^2_{max} and q^2_{min}. For $\theta \ll 1$ we have, from eq. (16),

$$E' = \frac{A^2 + m_e^2}{2A} + \frac{(A^2 - m_e^2)^2}{8A^3}\theta^2 + \mathcal{O}(\theta^4), \tag{22}$$

where

$$A = E(1+\beta)(1-y), \tag{23}$$

and eq. (15) becomes

$$q^2 = -\frac{m_e^2 y^2}{1-y} - \frac{E(1+\beta)(A^2 - m_e^2)^2}{4A^3}\theta^2 + \mathcal{O}(\theta^4). \tag{24}$$

The value of q^2_{max} is obtained by taking $\theta = 0$, namely

$$q^2_{max} = -\frac{m_e^2 y^2}{1-y}. \tag{25}$$

Analogously, the value of q^2_{min} is obtained when θ equals its maximum value θ_c. If $\theta_c \ll 1$ we can use eq. (24) to obtain

$$\begin{aligned} q^2_{min} &= -\frac{m_e^2 y^2}{1-y} - \frac{E(1+\beta)(A^2-m_e^2)^2}{4A^3}\theta_c^2 + \mathcal{O}(\theta^4) \\ &= -\frac{m_e^2 y^2}{1-y} - E^2(1-y)\theta_c^2 + \mathcal{O}(E^2\theta_c^4, m_e^2\theta_c^2, m_e^4/E^2). \end{aligned} \tag{26}$$

The function $f_\gamma^{(e)}(y)$ takes the form

$$\begin{aligned} f_\gamma^{(e)}(y) &= \frac{\alpha_{\rm em}}{2\pi}\left\{2(1-y)\left[\frac{m_e^2 y}{E^2(1-y)^2\theta_c^2 + m_e^2 y^2} - \frac{1}{y}\right]\right. \\ &\left. + \frac{1+(1-y)^2}{y}\log\frac{E^2(1-y)^2\theta_c^2 + m_e^2 y^2}{m_e^2 y^2} + \mathcal{O}(\theta_c^2, m_e^2/E^2)\right\}. \end{aligned} \tag{27}$$

Observe that the uncertainty is small with respect to 1. Also notice that the non-logarithmic term is singular in y and therefore represents a non-negligible correction. This subleading term is universal, in the sense that it is independent of the particular hard process we are considering.

We now comment on the effect of the $\mathcal{O}(q^2)$ terms in eq. (8). It is straightforward to verify that these terms will add a contribution to the differential cross section given by

$$\Delta d\sigma_{ep} = \frac{\alpha_{\rm em}}{2\pi}\frac{1}{2k\cdot q}\left[\sum_{n=0} A^{(n)}(k\cdot q)\left(\frac{q^2}{q\cdot k}\right)^n \frac{dq^2}{q\cdot k}\right]\frac{dy}{y}, \tag{28}$$

where the coefficients $A^{(n)}(q\cdot k)$ are obtained from the derivatives of the W_2 form factor evaluated at $q^2 = 0$, and are of order 1. For simplicity we neglected terms that are finite for $y \to 0$, proportional to derivatives of both W_1 and W_2, and for which the following conclusions hold as well. Upon integration over q^2, we are left with

$$\Delta d\sigma_{ep} = \frac{\alpha_{\rm em}}{2\pi}\frac{1}{2k\cdot q}\left[\sum_{n=1}\frac{A^{(n-1)}}{n}\left(\frac{q^2_{min}}{q\cdot k}\right)^n\right]\frac{dy}{y}, \tag{29}$$

where we neglected the contributions proportional to q^2_{max}, which are suppressed by powers of m_e^2/s. In the case of a small angular cut we have

$$\frac{|q^2_{min}|}{2q\cdot k} < \frac{E^2\theta_c^2}{S_{min}}. \tag{30}$$

For example, in the case of charm production and using $E = 30$ GeV, $\theta_c = 5 \times 10^{-3}$ we would have $|q^2_{min}|/(2q \cdot k) < 2.5 \times 10^{-3}$. Therefore these non-factorisable corrections are negligible, at least formally, with respect to the factorisable ones.

If no angular cut is applied, however, the ratio $|q^2_{min}|/(2q \cdot k)$, although limited from above by 1, can be of order 1. In this case, non-factorisable corrections to the Weizsäcker-Williams approximation are of order $1/y$.

As an application of the formulae discussed above, we consider the case of heavy-quark electroproduction. The total cross section for this process has recently been calculated at next-to-leading order in QCD by E. Laenen et al. in ref. [4]. For simplicity we will perform our comparisons between exact and approximated results at the leading order in α_S. For our goals this is not a limitation, as higher-order QCD corrections should not affect the properties of the photon density inside the electron and therefore will not change our results qualitatively.

In table 1 we show the total production cross section of charm and bottom quark pairs at HERA ($E = 30$ GeV, $\sqrt{S} = 314$ GeV), with the application of different angular cuts. We used $m_c = 1.5$ GeV, $m_b = 4.75$ GeV, and the MRSD0 parton distributions [5] for the proton. The matrix elements for the exact leading-order calculation are taken from ref. [4].

As can be seen from table 1, the presence of the non-logarithmic term improves the agreement between the exact result and the Weizsäcker-Williams approximation from the level of 6–7% to the level of 1% in the case of charm, and from 5–7% to 0.1% the case of the bottom (apart from the case $\theta_c = 0.5$, which is too large for the small-angle approximation to work properly).

Table 2 is similar, but evaluated for a configuration with electron beam energy $E = 100$ GeV and $\sqrt{S}$=1 TeV. We see that with the same angular cuts but higher energy, the approximation becomes worse (this can easily be explained from eq. (30)).

Let us now consider the case when no angular cut is imposed on the scattered electron, and the angular integration is performed over the whole phase space. In this case, the minimum value of the photon virtuality q^2_{min} can be computed in a simple way by imposing that the invariant mass of the produced hadronic system be bounded from below:

$$(q + k)^2 \geq S_{min}, \tag{31}$$

where, for example, $S_{min} = 4M^2$ in the case of heavy-quark pair production. Equa-

	Charm				
θ_c	$\sigma_{\rm exact}$ (nb)	σ_{WW} only log (nb)	(%)	σ_{WW} (nb)	(%)
5×10^{-1}	353.3	420.7	19.08	405.5	14.77
5×10^{-2}	325.4	345.2	6.08	330.0	1.41
5×10^{-3}	252.9	269.8	6.68	254.6	0.67
	Bottom				
θ_c	$\sigma_{\rm exact}$ (nb)	σ_{WW} only log (nb)	(%)	σ_{WW} (nb)	(%)
5×10^{-1}	3.110	3.404	9.45	3.277	5.37
5×10^{-2}	2.599	2.732	5.12	2.605	0.23
5×10^{-3}	1.931	2.060	6.68	1.933	0.10

Table 1: Total cross sections for charm and bottom production at HERA ($E = 30$ GeV, $\sqrt{S} = 314$ GeV) for various angular cuts. The percentual differences between the exact results and the two approximate ones are also shown. Statistical errors are smaller that the last digit.

	Charm				
θ_c	$\sigma_{\rm exact}$ (nb)	σ_{WW} only log (nb)	(%)	σ_{WW} (nb)	(%)
5×10^{-1}	660.8	866.4	31.11	839.0	26.97
5×10^{-2}	650.7	732.7	12.60	705.3	8.39
5×10^{-3}	549.6	599.0	8.99	571.6	4.00
	Bottom				
θ_c	$\sigma_{\rm exact}$ (nb)	σ_{WW} only log (nb)	(%)	σ_{WW} (nb)	(%)
5×10^{-1}	8.585	10.20	18.81	9.866	14.92
5×10^{-2}	7.997	8.492	6.19	8.154	1.96
5×10^{-3}	6.360	6.780	6.60	6.441	1.27

Table 2: Total cross sections for charm and bottom production at $E = 100$ GeV and $\sqrt{S} = 1$ TeV for various angular cuts. The percentual differences between the exact results and the two approximate ones are also shown. Statistical errors are smaller that the last digit.

tion (31) gives

$$q^2_{min} = -2k \cdot q + 4M^2 = -2k \cdot py + 4M^2 \simeq -(ys - 4M^2). \tag{32}$$

The integration limits become

$$-\left(ys - 4M^2\right) \leq q^2 \leq -\frac{m_e^2 y^2}{1-y}. \tag{33}$$

In this case, however, we cannot simply take $q^2 = 0$ in $W_1(q^2, k \cdot q)$, as we did in the previous case. In fact, processes involving the absorption of a virtual photon by a hadron target are characterized by a dimensional parameter λ, defined in such a way that the virtual photon cross section is nearly equal to the photoproduction cross section for $|q^2| < \lambda^2$, while it falls rapidly to zero for virtualities above this value. In particular, for the production of heavy objects, one can assume $\lambda = \xi M$, where ξ is a dimensionless parameter of order 1, since the characteristic scale of the process is given by the mass of the heavy object. For this reason, with quite a drastic approximation, instead of eq. (21) we may assume

$$\frac{W_1(q^2, k \cdot q)}{4k \cdot q} \simeq \sigma_{\gamma p}(q, k) \Theta\left(q^2 + \xi^2 M^2\right), \tag{34}$$

which leads to

$$f_\gamma^{(e)}(y) = \frac{\alpha_{\rm em}}{2\pi} \left[2m_e^2 y \left(\frac{1}{Q^2_{WW}} - \frac{1-y}{m_e^2 y^2}\right) + \frac{1+(1-y)^2}{y} \log \frac{Q^2_{WW}(1-y)}{m_e^2 y^2}\right], \tag{35}$$

where the quantity

$$Q^2_{WW} \equiv \min(ys - 4M^2, \xi^2 M^2) \tag{36}$$

is usually referred to as the effective Weizsäcker-Williams scale.

In practice, $ys - 4M^2$ will also be of order M^2, because of the competing effects of the large-x suppression of the gluon density of the proton (which favours the threshold region) and dynamical threshold suppression. We therefore expect that in general the approximation

$$Q^2_{WW} = \xi^2 M^2 \tag{37}$$

will be appropriate. We see that, at least for the production of heavy objects, the effective Weizsäcker-Williams scale is determined by dynamical rather than kinematical considerations. In the presence of an angular cut, dynamical arguments are ineffective, because for small angles (which are of interest in photoproduction processes) the minimum value of the virtuality, eq. (26), is well below $\xi^2 M^2$ in absolute value.

The parameter ξ is essentially unconstrained (except that we expect it to be of order 1). One can therefore absorb the non-logarithmic term in eq. (35) by an appropriate redefinition of ξ. Consider for example eq. (35) with only the most singular terms in y taken into account. We can get rid of the non-logarithmic term by replacing ξ with ξ', where ξ' is determined by the condition

$$-\frac{2}{y}+\frac{2}{y}\log\frac{\xi^2 M^2}{m_e^2 y^2}=\frac{2}{y}\log\frac{\xi'^2 M^2}{m_e^2 y^2}, \tag{38}$$

or

$$\xi' \simeq 0.6\,\xi. \tag{39}$$

The numerical results for total cross sections at HERA, both for charm and bottom production, are presented in table 3. In the first row we show the result of the exact leading-order calculation. We see that the use of the usual Weizsäcker-Williams approximation of eq. (1), shown in the second row, leads to a completely wrong result. The following two entries are obtained using eq. (35) with $Q^2_{WW}=ys-4M^2$. The effect of the inclusion of the non-logarithmic term is apparent, but still the approximation is not satisfactory. The remaining entries are obtained by implementing the dynamical considerations made above. We performed the calculation using only the logarithmic term in the Weizsäcker-Williams distribution, and choosing different values for ξ, namely $\xi = 1/2, 1, 2$. The choice $\xi = 1$ gives a very good agreement between the exact and the approximated result. We then tested eq. (39) by including the non-logarithmic term in the Weizsäcker-Williams function and taking $Q_{WW} = M/0.6$. Finally, the comparison between the entries in the fifth and the last rows is a test of eq. (37).

	Charm		Bottom	
Q_{WW}	σ (nb)	(%)	σ (nb)	(%)
Exact result	353.4	-	3.135	-
$\sqrt{yS-4M^2}$ only log	478.4	35.37	4.090	30.46
$\sqrt{ys-4M^2}$ only log	383.1	8.40	3.392	8.20
$\sqrt{ys-4M^2}$	367.9	4.10	3.265	4.15
M only log	349.9	0.99	3.137	0.06
$M/2$ only log	327.1	7.44	2.934	6.41
$2M$ only log	372.6	5.43	3.339	6.51
$M/0.6$	351.4	0.57	3.158	0.73
$min(\sqrt{ys-4M^2}, M)$ only log	349.0	1.24	3.128	0.22

Table 3: Total cross sections for heavy-flavour production at HERA, for different choices of the Weizsäcker-Williams distribution. The percentual differences between the exact results and the approximate ones are also shown.

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